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The Lebesgue Constants for Some Cardinal Spline Interpolation Operators

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1. INTRODUCTION

For $n = 1, 2, 3,...$ and $1 \leqslant r \leqslant n$, we define

$$\mathscr{S}_{n,r} = \{S \in C^{n-r-1}(\mathbb{R}): S(x) \text{ is bounded on } \mathbb{R} \text{ and on } (\nu, \nu+1) \text{ it equals a polynomial of degree } <n, \forall\, \nu \in \mathbb{Z}\}.$$

Now take points $0 \leqslant \alpha_1 < \alpha_2 < \cdots < \alpha_r < 1$, where $\alpha_1 = 0$ if $n + r$ is odd, $\alpha_1 > 0$ otherwise, and the non-zero α_j are symmetric about $\frac{1}{2}$.

Suppose that for $s = 1, 2,..., r$, we have

$$\mathbf{y}^{(s)} = (y_\nu^{(s)})_{\nu=-\infty}^{\infty} \in l_\infty \text{ and } \mathbf{y} = (\mathbf{y}^{(1)}, \mathbf{y}^{(2)},..., \mathbf{y}^{(r)}). \tag{1}$$

Then it follows from the work of Micchelli [3] that there is a unique element $\mathscr{L}_{n,r}\mathbf{y}$ in $\mathscr{S}_{n,r}$ such that

$$\mathscr{L}_{n,r}\, y(\nu + \alpha_s) = y_\nu^{(s)}, \qquad \nu \in \mathbb{Z},\ s = 1, 2,..., r.$$

We define $\|\mathscr{L}_{n,r}\| = \sup\{\|\mathscr{L}_{n,r}\mathbf{y}\|: \|\mathbf{y}\|_\infty = 1\}$, the "$n$th Lebesgue constant" for the interpolation considered. For $r = n$ or $n - 1$, the above interpolation reduces to polynomial interpolation. In these cases Erdös [1] has shown there exists a constant c such that

$$\|\mathscr{L}_{n,r}\| \geqslant \frac{2}{\pi} \log n - c, \tag{2}$$

for any choice of $\alpha_1,..., \alpha_r$.

Moreover Rivlin [5] has shown that if $r = n$ and

$$\alpha_j = \cos\frac{(2j-1)\pi}{2n}, \qquad j = 1, 2,..., n,$$

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then

$$\|\mathscr{L}_{n,n}\| = \frac{2}{\pi} \log n + \frac{2}{\pi} \left(\log \frac{8}{\pi} + \gamma \right) + o(1), \tag{3}$$

where γ is the Euler–Mascheroni constant.

For $r = 1$, the above interpolation reduces to ordinary cardinal spline interpolation and for this case Richards [4] has shown that

$$\|\mathscr{L}_{n,1}\| = \frac{2}{\pi} \log n + \frac{2}{\pi} \left(2 \log \frac{4}{\pi} + \gamma \right) + o(1). \tag{4}$$

In this paper we prove the following result, which reduces to (4) when $r = 1$. Much of our proof follows the approach of Richards in [4].

THEOREM 1. *For fixed points* $0 \leqslant \alpha_1 < \alpha_2 < \cdots < \alpha_r < 1$, *with the non-zero* α_j *symmetric about* $\frac{1}{2}$, *there are constants* M_1, M_2 *such that*

$$\|\mathscr{L}_{n,r}\| = M_1 \operatorname{Log} n + M_2 + o(1), \tag{5}$$

where n is restricted to have the same parity as r if $\alpha_1 > 0$ *and different parity from r if* $\alpha_1 = 0$. *Moreover* $M_1 \geqslant 2/\pi$ *with equality iff*

$$\alpha_j = \frac{2j-1}{2r}, \qquad j = 1, \ldots, r,$$

or

$$\alpha_j = \frac{j-1}{r}, \qquad j = 1, \ldots, r.$$

In these cases,

$$M_2 = \frac{2}{\pi} \left(2 \log \frac{4}{\pi} + \gamma \right). \tag{6}$$

2. PRELIMINARIES

Fix points $0 \leqslant \alpha_1 < \alpha_2 < \cdots < \alpha_r < 1$ with the non-zero α_j symmetric about $\frac{1}{2}$ and take $n \geqslant r$, where $n + r$ is odd iff $\alpha_1 = 0$.

For any x in $\mathbb{R}$, $-\pi < u < \pi$, $u \neq 0$, and $1 \leqslant s \leqslant r$, we define

$$\Omega_s(x, u) = \frac{\det(\sum (e^{\pi i k \beta_l}/(u + \pi k)^{n-m+1}))_{l,m=1}^r}{\det(\sum (e^{\pi i k \alpha_l}/(u + \pi k)^{n-m+1}))_{l,m=1}^r}, \tag{7}$$

where the summations are taken over all k of the same parity as r and

$$\begin{aligned} \beta_l &= \alpha_l, \qquad l \neq s, \\ &= x, \qquad l = s. \end{aligned}$$

Next define $S(x; u) = e^{iux}\Omega_s(x, u)$.

These definitions are generalisations of work of Schoenberg in [6]. For taking even n and $r = 1$ we have

$$\begin{aligned} S(x; u) &= e^{iux} \frac{\sum_{k=-\infty}^{\infty} (e^{\pi i(2k+1)x}/(u + (2k+1)\pi)^n)}{\sum_{k=-\infty}^{\infty} (u + (2k+1)\pi)^{-n}} \\ &= e^{iux} \frac{\sum_{k=-\infty}^{\infty} (e^{2\pi ikx}/(v + 2k\pi)^n)}{\sum_{k=-\infty}^{\infty} (v + 2k\pi)^{-n}}, \\ &\text{where} \quad v = u + \pi, \end{aligned}$$

and this is precisely Schoenberg's exponential Euler spline $S_{n-1}(x; e^{iv})$.

In general $S(x; u)$ is a linear combination of exponential Euler splines of degrees $n-1, n-2, \ldots, n-r$ and so $S(-; u)$ is in $\mathscr{S}_{n,r}$.

Now $\Omega_s(x, u)$ is a continuous function of x and u, $u \neq 0$, and $|\Omega_s(x, u)| = |S(x; u)| \leqslant \|\mathscr{L}_{n,r}\|$. Define

$$L_s(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{iu(x-\alpha_s)} \Omega_s(x, u)\, du. \tag{8}$$

Then L_s is in $\mathscr{S}_{n,r}$ and $L_s(\alpha_j + \nu) = \delta_{js}\delta_{\nu 0}$, ν in $\mathbb{Z}$, $j = 1, 2, \ldots, r$.

We shall be interested in the behaviour of $\Omega_s(x, u)$ as $n \to \infty$. Now if r is even,

$$\begin{aligned} \Omega_s(x, u) &= \frac{\det(\sum_{k=-\infty}^{\infty} (e^{2\pi ik\beta_l}/(u + 2k\pi)^{n-m+1}))_{l,m=1}^r}{\det(\sum_{k=-\infty}^{\infty} (e^{2\pi ik\alpha_l}/(u + 2k\pi)^{n-m+1}))_{l,m=1}^r} \\ &= \frac{\sum_{k_1,\ldots,k_r} \{V(k_1,\ldots,k_r) \prod_{j=1}^r (e^{2\pi ik_j\beta_j}/(u + 2k_j\pi)^n)\}}{\sum_{k_1,\ldots,k_r} \{V(k_1,\ldots,k_r) \prod_{j=1}^r (e^{2\pi ik_j\alpha_j}/(u + 2k_j\pi)^n)\}}, \end{aligned}$$

where for any numbers $a_1, \ldots, a_r$, we denote by $V(a_1, \ldots, a_r)$ the Vandermande determinant

$$\det(a_m^{l-1})_{l,m=1}^r = \prod_{1 \leqslant j < k \leqslant r} (a_k - a_j).$$

Hence

$$\Omega_s(x,u)=\frac{\sum_{k_1<\cdots<k_r}\{V(k_1,\dots,k_r)\,N(k_1,\dots,k_r)\prod_{j=1}^r(u+2\pi k_j)^{-n}\}}{\sum_{k_1<\cdots<k_r}\{V(k_1,\dots,k_r)\,D(k_1,\dots,k_r)\prod_{j=1}^r(u+2\pi k_j)^{-n}\}}, \tag{9}$$

where $N(k_1,\dots,k_r)=\det(e^{2\pi i k_m\beta_l})_{l,m=1}^r$ and $D(k_1,\dots,k_r)=\det(e^{2\pi i k_m\alpha_l})_{l,m=1}^r$.

We might expect that as $n\to\infty$, each summation in (9) is dominated by the largest two terms and thus $\Omega_s(x,u)$ is close to

$$\hat{\Omega}_s(x,u)=\frac{\begin{bmatrix}V(-r/2,\dots,r/2-1)\,N(-r/2,\dots,r/2-1)(u+r\pi)^n\\ \quad+V(-r/2+1,\dots,r/2)\,N(-r/2+1,\dots,r/2)(u-r\pi)^n\end{bmatrix}}{\begin{bmatrix}V(-r/2,\dots,r/2-1)\,D(-r/2,\dots,r/2-1)(u+r\pi)^n\\ \quad+V(-r/2+1,\dots,r/2)\,D(-r/2+1,\dots,r/2)(u-r\pi)^n\end{bmatrix}}.$$

After simplification we find

$$\operatorname{Im}\hat{\Omega}_s(x,u)=\frac{A_s(x)\{(r\pi+u)^n-(r\pi-u)^n\}}{V\{(r\pi+u)^n+(r\pi-u)^n\}} \tag{10}$$

and

$$\operatorname{Re}\hat{\Omega}_s(x,u)=B_s(x)/V, \tag{11}$$

where

$$A_s(x)=e^{(r-1)\pi i(\alpha_s-x)}\sin\pi(\alpha_s-x)\,V(e^{2\pi i\beta_1},\dots,e^{2\pi i\beta_r}), \tag{12}$$

and

$$B_s(x)=e^{(r-1)\pi i(\alpha_s-x)}\cos\pi(\alpha_s-x)\,V(e^{2\pi i\beta_1},\dots,e^{2\pi i\beta_r}), \tag{13}$$

and

$$V=V(e^{2\pi i\alpha_1},\dots,e^{2\pi i\alpha_r}). \tag{14}$$

A similar calculation for odd r produces the same formulae (10) to (14).

The following lemma states in what sense $\Omega_s(x,u)$ is close to $\hat{\Omega}_s(x,u)$ for large n.

LEMMA 1. *If* $x\neq\alpha_j$, $j=1,\dots,r$, *then* $\{\operatorname{Re}\Omega_s(x,u)-\operatorname{Re}\hat{\Omega}_s(x,u)\}$ *and* $u^{-1}\{\operatorname{Im}\Omega_s(x,u)-\operatorname{Im}\hat{\Omega}_s(x,u)\}$ *both converge to zero uniformly in* u *on* $(0,\pi)$ *as* $n\to\infty$.

Proof. We give the proof for even r, the case of odd r following similarly. For simplicity we write $u = 2\pi v$. Then from (9), $\Omega_s(x, u) = N/D$, where

$$\begin{aligned} N = {} & V\left(-\frac{r}{2},..., \frac{r}{2} - 1\right) N\left(-\frac{r}{2},..., \frac{r}{2} - 1\right) \\ & + V\left(-\frac{r}{2} + 1,..., \frac{r}{2}\right) N\left(-\frac{r}{2} + 1,..., \frac{r}{2}\right) \frac{(v - r/2)^n}{(v + r/2)^n} \\ & + \sum V(k_1,..., k_r) N(k_1,..., k_r) \frac{(v - r/2)^n \cdots (v + r/2 - 1)^n}{(v + k_1)^n \cdots (v + k_r)^n}, \end{aligned} \tag{15}$$

D is the same as N with $N(k_1,..., k_r)$ replaced throughout by $D(k_1,..., k_r)$, and the summations are taken over all $(k_1,..., k_r)$ with $k_1 < \cdots < k_r$ and not equal to $(-r/2,..., r/2 - 1)$ or $(-r/2 + 1,..., r/2)$.

We shall show

$$X_n(v) = \sum \left| V(k_1,..., k_r) \frac{(v - r/2)^n \cdots (v + r/2 - 1)^n}{(v + k_1)^n \cdots (v + k_r)^n} \right| \tag{16}$$

(with summation as in (15)) converges uniformly to zero on $(0, \frac{1}{2})$ as $n \to \infty$. Since $|N(k_1,..., k_r)|, |D(k_1,..., k_r)| \leqslant r!$, this implies $\Omega_s(x, u)$ converges uniformly to $\hat{\Omega}_s(x, u)$ on $(0, \pi)$ as $n \to \infty$.

Now for $k \geqslant 0$ and $l \geqslant k$ or $l \leqslant -k - 1$,

$$\left(\frac{v + k}{v + l}\right)^n \leqslant \left(\frac{2k + 1}{2l + 1}\right)^n, \qquad \forall v \in [0, \tfrac{1}{2}],$$

and for $k < 0$ and $|l| \geqslant |k|$,

$$\left(\frac{v + k}{v + l}\right)^n \leqslant \left(\frac{k}{l}\right)^n, \qquad \forall v \in [0, \tfrac{1}{2}].$$

So for large enough n,

$$\begin{aligned} X_n(v) & \leqslant [\max_{1 \leqslant i < j \leqslant r} |k_j - k_i|]^{(1/2)r(r-1)} \sum \left| \frac{(v - r/2)^n \cdots (v + r/2 - 1)^n}{(v + k_1)^n \cdots (v + k_r)^n} \right| \\ & \leqslant 2^r \left\{ \sum_{\substack{l=1 \\ l \text{ odd}}}^{\infty} \frac{l^{(1/2)r(r-1)}}{l^n} \sum_{\substack{l=2 \\ l \text{ even}}}^{\infty} l^{(1/2)r(r-1)} \left(\frac{2}{l}\right)^n \sum_{\substack{l=3 \\ l \text{ odd}}}^{\infty} l^{1/2r(r-1)} \left(\frac{3}{l}\right)^n \cdots \right. \\ & \qquad \left. \sum_{\substack{l=r \\ l \text{ even}}}^{\infty} l^{(1/2)r(r-1)} \left(\frac{r}{l}\right)^n - (r!)^{(1/2)r(r-1)} \right\} \end{aligned}$$

$$= 2^r (r!)^{(1/2)r(r-1)} \left\{ \sum_{\substack{l=1 \\ l \text{ odd}}}^{\infty} l^{(1/2)r(r-1)-n} \sum_{\substack{l=2 \\ l \text{ even}}}^{\infty} \left(\frac{l}{2}\right)^{(1/2)r(r-1)-n} \right.$$

$$\left. \times \sum_{\substack{l=3 \\ l \text{ odd}}}^{\infty} \left(\frac{l}{3}\right)^{(1/2)r(r-1)-n} \cdots \sum_{\substack{l=r \\ l \text{ even}}}^{\infty} \left(\frac{l}{r}\right)^{(1/2)r(r-1)-n} - 1 \right\}$$

$$\leqslant 2^r (r!)^{(1/2)r(r-1)} \left\{ \left[\sum_{\substack{l=r \\ l \text{ even}}}^{\infty} \left(\frac{l}{r}\right)^{(1/2)r(r-1)-n} \right]^r - 1 \right\}$$

$$\leqslant 2^r (r!)^{(1/2)r(r-1)} \left\{ \left[1 + r\left(\frac{r}{r+2}\right)^{n-(1/2)r(r-1)-1} \right]^r - 1 \right\}$$

$$\leqslant 2^{2r} (r!)^{(1/2)r(r-1)} r \left(\frac{r}{r+1}\right)^{n-(1/2)r(r-1)-1}. \tag{17}$$

So $X_n(v)$ converges uniformly to zero on $(0, \frac{1}{2})$ as $n \to \infty$. Now noting that $\overline{N(k_1, ..., k_r)} = N(-k_1, ..., -k_r)$ and $\overline{D(k_1, ..., k_r)} = D(-k_1, ..., -k_r) = (-1)^{(1/2)r+n} D(k_1, ..., k_r)$, we have from (9) that

$\operatorname{Im} \Omega_s(x, u)$

$$= \frac{\left[\begin{array}{l} \sum_{k_1 < \cdots < k_r} V(k_1, ..., k_r) \operatorname{Im}[i^{(1/2)r+n} N(k_1, ..., k_r)] \\ \quad \times \{\prod_{j=1}^r (v + k_j)^{-n} - (-1)^n \prod_{j=1}^r (v - k_j)^{-n}\} \end{array} \right]}{\left[\begin{array}{l} \sum_{k_1 < \cdots < k_r} V(k_1, ..., k_r)\, i^{(1/2)r+n} D(k_1, ..., k_r) \\ \quad \times \{\prod_{j=1}^r (v + k_j)^{-n} + (-1)^n \prod_{j=1}^r (v - k_j)^{-n}\} \end{array} \right]}.$$

Thus to show $u^{-1}\{\operatorname{Im} \Omega_s(x, u) - \operatorname{Im} \hat{\Omega}_s(x, u)\}$ converges to zero uniformly on $(0, \pi)$ as $n \to \infty$, it is sufficient to show

$$Y_n(v) = v^{-1} \sum \left| V(k_1, ..., k_r) \times \left\{ \prod_{j=1}^r (v + k_j)^{-n} - (-1)^n \prod_{j=1}^r (v - k_j)^{-n} \right\} \times \left(v - \frac{r}{2}\right)^n \cdots \left(v + \frac{r}{2} - 1\right)^n \right|$$

(with summation as in (15)) converges uniformly to zero on $(0, \frac{1}{2})$ as $n \to \infty$. We shall prove this for even n, the case of odd n following similarly.

Now assuming $(v + k_1)^n \cdots (v + k_r)^n < (v - k_1)^n \cdots (v - k_r)^n$ we have

$$\frac{1}{v}\left\{1-\frac{(v+k_1)^n\cdots(v+k_r)^n}{(v-k_1)^n\cdots(v-k_r)^n}\right\}$$

$$\leqslant\frac{n}{v}\left\{1-\frac{|v+l_1|\cdots|+l_t|}{|v-l_1|\cdots|v-l_t|}\right\}$$

(where $l_1,\ldots,l_t$ are the non-zero $k_1,\ldots,k_r$)

$$\leqslant\frac{n}{v}\left\{\frac{(l_1-v)\cdots(l_t-v)-(l_1+v)\cdots(l_t+v)}{(l_1-v)\cdots(l_t-v)}\right\}$$

$$\leqslant\frac{2n(|l_1|+1)\cdots(|l_t|+1)}{|l_1-v|\cdots|l_t-v|}$$

$$\leqslant 2n4^r.$$

Thus $Y_n(v)\leqslant n4^{r+1}X_n(v)$ and its follows from (17) that $Y_n(v)$ converges to zero uniformly on $(0,\frac{1}{2})$ as $n\to\infty$. ∎

Now it follows from the work of [3] that for any y as in (1), and L_s defined as in (8),

$$(\mathscr{L}_{n,r}\mathbf{y})(x)=\sum_{s=1}^{r}\sum_{v=-\infty}^{\infty}y_v^{(s)}L_s(x-v).$$

So

$$\|\mathscr{L}_{n,r}\|=\max_{0\leqslant x<1}\sum_{s=1}^{r}\sum_{v=-\infty}^{\infty}|L_s(x-v)|. \tag{18}$$

We therefore proceed to examine the sign of $L_s(x)$, using a method similar to that of Lipow in [2].

If $f\in\mathscr{S}_{n,r}$ is periodic with period P, we let $Z(f)$, the number of zeros of f in $[0,P)$, where zeros are counted according to multiplicity, an interval on which f vanishes is counted as a zero of multiplicity n, and a jump through zero is counted as a zero of multiplicity one.

LEMMA 2. *If $f\in\mathscr{S}_{n,r}$ has integral period P, then*

$$Z(f)\leqslant Pr, \qquad \textit{if } Pr \textit{ is even},$$
$$\leqslant Pr-1, \qquad \textit{if } Pr \textit{ is odd}.$$

Proof. It follows from Rolle's theorem that $Z(f)\leqslant Z(f')\leqslant\cdots\leqslant Z(f^{(n-r)})$. But $f^{(n-r)}$ is a polynomial of degree $r-1$ on each interval $(v,v+1)$ and so $Z(f^{(n+r)})\leqslant Pr$ with strict inequality if Pr is odd. ∎

LEMMA 3. *The zeros of* L_s *are simple and occur only at* $\alpha_j + \nu$, $j = 1,\ldots, r$, $\nu \in \mathbb{Z}$, *except when* $\nu = 0$ *and* $j = s$.

Proof. We give the proof for even n and r, the other cases following similarly. For $m = 1, 2,\ldots$, let $L_{s,m} \in \mathscr{S}_{n,r}$ satisfy

$$L_{s,m}(2km + \alpha_s) = 1, \qquad \forall\, k \in \mathbb{Z},$$

$$L_{s,m}((2k+1)m + \alpha_{r+1-s}) = -1, \qquad \forall\, k \in \mathbb{Z},$$

and $L_{s,m}(\nu + \alpha_j) = 0$, for all other $\nu \in \mathbb{Z}$ and $\alpha_1,\ldots, \alpha_r$.

Then $L_{s,m}(x)$ is antisymmetric about $x = \frac{1}{2}(m+1)$ and $x = \frac{1}{2}(3m+1)$. Also $L_{s,m}(\nu + \alpha_j) = 0$ for all $\nu = 0,\ldots, 2m-1$ and $j = 1,\ldots, r$ except for $\nu = 0$, $j = s$ and $\nu = m$, $j = r+1-s$. Since $L_{s,m}$ is periodic of period $2m$, Lemma 1 tells us that these are the only zeros of $L_{s,m}$.

Now $L_{s,m}(x) = \sum_{k=-\infty}^{\infty} L_s(x - 2km) - \sum_{k=-\infty}^{\infty} L_{r+1-s}(x - (2k+1)m)$ and so $|L_{s,m}(x) - L_s(x)| \leqslant \sum_{k \neq 0} |L_s(x - 2km)| + \sum_{k=-\infty}^{\infty} |L_{r+1-s}(x - (2k+1)m)|$. It follows from the work of [3] that $L_s(x)$ and $L_{r+1-s}(x)$ decay exponentially and thus $L_{s,m}(x)$ converges locally uniformly to $L_s(x)$ as $m \to \infty$. The result follows. ∎

3. PROOF OF THEOREM 1

Fix x with $\alpha_{k-1} < x < \alpha_k$ for some $1 \leqslant k \leqslant r$, where $\alpha_0 = \alpha_r - 1$. Then it follows from Lemma 3 that for $s = 1,\ldots, k-1$,

$$\begin{aligned}\operatorname{sgn} L_s(x - \nu) &= (-1)^{s+k+r\nu}, \qquad &\nu = 1, 2, 3,\ldots,\\ &= (-1)^{s+k+r\nu+1}, \qquad &\nu = 0, -1, -2,\ldots\end{aligned}$$

and for $s = k,\ldots, r$,

$$\begin{aligned}\operatorname{sgn} L_s(x - \nu) &= (-1)^{s+k+r\nu}, \qquad &\nu = 0, 1, 2,\ldots,\\ &= (-1)^{s+k+r\nu+1}, \qquad &\nu = -1, -2, -3,\ldots .\end{aligned}$$

Thus, if $s = 1,\ldots, k-1$,

$$\begin{aligned}&\sum_{\nu=-N+1}^{N} |L_s(x - \nu)|\\ &\quad = \frac{(-1)^{s+k}}{2\pi} \int_{-\pi}^{\pi} e^{iu(x-\alpha_s)}\, \Omega_s(x, u) \left\{ \sum_{\nu=1}^{N} e^{-iu\nu} - \sum_{\nu=-N+1}^{0} e^{-iu\nu} \right\} du\end{aligned}$$

$$= \frac{(-1)^{s+k}}{2\pi i} \int_{-\pi}^{\pi} e^{iu(x-\alpha_s-1/2)}\, \Omega_s(x,u)(1-\cos Nu) \operatorname{cosec} \frac{u}{2}\, du$$

$$= \frac{(-1)^{s+k}}{2\pi} \int_{-\pi}^{\pi} \operatorname{Im}\{e^{iu(x-\alpha_s-1/2)} \Omega_s(x,u)\}(1-\cos Nu) \operatorname{cosec} \frac{u}{2}\, du.$$

Similarly if $s = k,\ldots, r$,

$$\sum_{v=-N}^{N-1} |L_s(x-v)|$$

$$= \frac{(-1)^{s+k}}{2\pi} \int_{-\pi}^{\pi} \operatorname{Im}\, \{e^{iu(x-\alpha_s+1/2)} \Omega_s(x,u)\}(1-\cos Nu) \operatorname{cosec} \frac{u}{2}\, du.$$

Now it follows from Lemma 1 that for large enough n, $\operatorname{Im} \Omega_s(x,u) = O(u)$ as $u \to 0$ and it follows from the Riemann–Lebesgue Lemma that

$$\sum_{v=-\infty}^{\infty} |L_s(x-v)|$$

$$= \frac{(-1)^{s+k}}{2\pi} \int_{-\pi}^{\pi} \operatorname{Im}\{e^{iu(x-\alpha_s+\delta)} \Omega_s(x,u)\} \operatorname{cosec} \frac{u}{2}\, du,$$

where

$$\delta = \delta_s = -\tfrac{1}{2}, \qquad s = 1,\ldots, k-1,$$
$$= \tfrac{1}{2}, \qquad s = k,\ldots, r.$$

So

$$\sum_{v=-\infty}^{\infty} |L_s(x-v)| = \frac{(-1)^{s+k}}{\pi} (I_{1,s} + J_s),$$

where

$$I_{1,s} = \int_0^{\pi} \sin u(x-\alpha_s+\delta) \operatorname{Re} \Omega_s(x,u) \operatorname{cosec} \frac{u}{2}\, du$$

and

$$J_s = \int_0^{\pi} \cos u(x-\alpha_s+\delta) \operatorname{Im} \Omega_s(x,u) \operatorname{cosec} \frac{u}{2}\, du.$$

Let

$$\hat{J}_s = \int_0^{\pi} \cos u(x-\alpha_s+\delta) \operatorname{Im} \hat{\Omega}_s(x,u) \operatorname{cosec} \frac{u}{2}\, du.$$

Then by Lemma 1, $J_s = \hat{J}_s + o(1)$. Let $\hat{J}_s = I_{2,s} + I_{3,s}$, where

$$\begin{aligned} I_{3,s} &= \int_0^\pi \operatorname{Im} \hat{\Omega}_s(x,u) \frac{2}{u}\, du \\ &= \frac{2A_s(x)}{V} \int_0^\pi \frac{(r\pi+u)^n - (r\pi-u)^n}{(r\pi+u)^n + (r\pi-u)^n} \frac{du}{u} \\ &= \frac{4A_s(x)}{V} \int_{(r-1)/(r+1)}^1 \frac{1-t^n}{1+t^n} \frac{dt}{1-t^2}, \qquad \text{putting} \quad t = \frac{r\pi-u}{r\pi+u}. \end{aligned}$$

Now it is shown in [4] that

$$\int_0^1 \frac{1-t^n}{1+t^n} \frac{dt}{1-t^2} = \frac{1}{2} \left\{ \log n + \log \frac{4}{\pi} + \gamma \right\} o(1).$$

Also

$$\begin{aligned} \int_0^{(r-1)/(r+1)} \frac{1-t^n}{1+t^n} \frac{dt}{1-t^2} &= \int_0^{(r-1)/(r+1)} \frac{dt}{1-t^2} + o(1) \\ &= \frac{1}{2} \log r + o(1). \end{aligned}$$

So

$$I_{3,s} = \frac{2A_s(x)}{V} \left\{ \log n + \log \frac{4}{\pi} + \gamma - \log r \right\} + o(1). \tag{19}$$

Now $I_{2,s} = \int_0^\pi \{\cos u(x - \alpha_s + \delta) \operatorname{Im} \hat{\Omega}_s(x,u) \operatorname{cosec}(u/2) - \operatorname{Im} \hat{\Omega}_s(x,u)(2/u)\}\, du$. Since $|\operatorname{Im} \hat{\Omega}_s(x,u)| \leqslant |A_s(x)|/V$, for all $n = 1, 2, 3,\ldots$ and $\lim_{n\to\infty} \operatorname{Im} \hat{\Omega}_s(x,u) = A_s(x)/V$, for all $u \in (0,\pi]$, we have

$$I_{2,s} = \frac{A_s(x)}{V} \int_0^\pi \left\{ \operatorname{cosec} \frac{u}{2} \cos u(x - \alpha_s + \delta) - \frac{2}{u} \right\} du + o(1). \tag{20}$$

Furthermore it follows from Lemma 1 that

$$I_{1,s} = \frac{B_s(x)}{V} \int_0^\pi \sin u(x - \alpha_s + \delta) \operatorname{cosec} \frac{u}{2}\, du + o(1). \tag{21}$$

So to sum up,

$$\sum_{\nu=-\infty}^{\infty} |L_s(x-\nu)| = \frac{(-1)^{s+k}}{\pi} (I_{1,s} + I_{2,s} + I_{3,s}) + o(1),$$

where $I_{1,s}$, $I_{2,s}$ and $I_{3,s}$ are given by (21), (20) and (19). Thus from (18),

$$\|\mathscr{L}_{n,r}\| = M_1 \log n + M_2 + o(1), \tag{5}$$

where

$$M_1 = \max_{0 \leqslant x < 1} \frac{2}{\pi} \sum_{s=1}^{r} (-1)^{s+k} \frac{A_s(x)}{V}. \tag{22}$$

For $j = 1,..., r$, let

$$\begin{aligned} \beta_j &= (2j-1)/2r, && \text{if} \quad n+r \text{ is even,} \\ &= (j-1)/r, && \text{if} \quad n+r \text{ is odd.} \end{aligned}$$

We claim

$$M_1 \geqslant \frac{2}{\pi} \qquad \text{with equality iff} \quad \alpha_j = \beta_j,\ j = 1,..., r. \tag{23}$$

We shall prove (23) only for even n and r, the other cases following similarly. Now for any x in $\mathbb{R}$, let

$$\begin{aligned} F(x) &= \sum_{s=1}^{r} (-1)^{s+1} \frac{A_s(x)}{V} \\ &= a_0 \cos r\pi x + a_1 \cos(r-2)\pi x + \cdots + a_{r/2}. \end{aligned}$$

Then

$$F(\alpha_i) = 0,\ i = 1,..., r.$$

Now

$$a_0 = \tilde{V}(e^{2\pi i \alpha_1},..., e^{2\pi i \alpha_r})/iV,$$

where $\tilde{V}(e^{2\pi i \alpha_1},..., e^{2\pi i \alpha_r})$ is $V(e^{2\pi i \alpha_1},..., e^{2\pi i \alpha_r})$ with the first row replaced by $\mathbf{y}$ with $\mathbf{y}_\nu = (-1)^{\nu+1} e^{r\pi i \alpha_\nu}$, $\nu = 1,..., r$. We shall show that

$$a_0 \geqslant 1 \qquad \text{with equality iff } \alpha_j = \beta_j,\ j = 1,..., r. \tag{24}$$

Expanding the determinants by the first row, we have

$$iVa_0 - iV = \sum_{s=1}^{r} [e^{r\pi i \alpha_s} + i(-1)^s]\, e^{-2\pi i \alpha_s} V(e^{2\pi i \alpha_1},..., \widehat{e^{2\pi i \alpha_s}},..., e^{2\pi i \alpha_r})$$

(where $\hat{}$ denotes that this term is to be omitted),

$$= 2 \sum_{s=1}^{r/2} \{1 + (-1)^s \sin r\pi\alpha_s\} \, e^{(r-2)\pi i\alpha_s} V(e^{2\pi i\alpha_1},..., \widehat{e^{2\pi i\alpha_s}},..., e^{2\pi i\alpha_r})$$

since

$$V(e^{2\pi i\alpha_1},..., \widehat{e^{2\pi i\alpha_{s+r+1}}},..., e^{2\pi i\alpha_r}) = e^{2\pi i(r-2)\alpha_s} V(e^{2\pi i\alpha_1},..., \widehat{e^{2\pi i\alpha_s}},..., e^{2\pi i\alpha_r}).$$

Now a straightforward calculation shows $i^{(1/2)r} V > 0$ and

$$i^{(1/2)r-1} e^{(r-2)\pi i\alpha_s} V(e^{2\pi i\alpha_1},..., \widehat{e^{2\pi i\alpha_s}},..., e^{2\pi i\alpha_r}) > 0, \; s = 1,..., r.$$

So $i^{(1/2)r} V(a_0 - 1) \geqslant 0$ with equality iff

$$1 + (-1)^s \sin r\pi\alpha_s = 0, \qquad s = 1,..., \tfrac{1}{2}r.$$

This gives (24) and hence

$$\max_{0 \leqslant x \leqslant 1} |F(x)| \geqslant 1 \qquad \text{with equality iff } F(x) = \cos r\pi x,$$

i.e., with equality iff $\alpha_j = \beta_j$, $j = 1,..., r$.

We have thus proved (23).

Henceforward we assume $\alpha_j = \beta_j$, $j = 1,..., r$. To complete the proof of Theorem 1 we must show M_2 is given by (6). Note that the maximum in (22) is attained for $x = \gamma_j$, $j = 1,..., r$, where

$$\gamma_j = (j-1)/r, \qquad \text{if} \quad n + r \text{ is even},$$
$$= (2j-1)/r, \qquad \text{if} \quad n + r \text{ is odd}.$$

So M_2 equals the maximum over $x = \gamma_j$, $j = 1,..., r$, of

$$\frac{1}{\pi} \sum_{s=1}^{r} (-1)^{s+k} \left\{ \frac{2A_s(x)}{V} \left(\log \frac{4}{2\pi} + \gamma \right) + I_s \right\}, \tag{25}$$

where

$$I_s = \frac{A_s(x)}{V} \int_0^\pi \left\{ \operatorname{cosec} \frac{u}{2} \cos u(x - \alpha_s + \delta) - \frac{2}{u} \right\} du$$
$$+ \frac{B_s(x)}{V} \int_0^\pi \sin u(x - \alpha_s + \delta) \operatorname{cosec} \frac{u}{2} \, du.$$

Now a straightforward calculation shows

$$A_s(\gamma_j) = \prod_{\nu=1}^{r} \sin \pi(\beta_\nu - \gamma_j) \Big/ \prod_{\substack{\nu=1 \\ \nu \neq s}}^{r} \sin \pi(\beta_\nu - \beta_s) = \frac{(-1)^{s+j+n+r}}{r}.$$

Thus we find (25) takes the same values for all $x = \gamma_j$, $j = 1, \ldots, r$, and

$$M_2 = \frac{2}{\pi}\left(\log\frac{4}{r\pi} + \gamma\right) + \frac{1}{\pi}\int_0^\pi \left\{f(u)\operatorname{cosec}\frac{u}{2} - \frac{2}{u}\right\} du, \tag{26}$$

where

$$\begin{aligned} f(u) = \frac{1}{r}\sum_{s=1}^{r} \bigg\{&\cos u\left(\frac{1}{2} - \frac{2s-1}{2r}\right) \\ &+ \sin u\left(\frac{1}{2} - \frac{2s-1}{2r}\right)\cot\frac{(2s-1)\pi}{2r}\bigg\} \\ = \frac{1}{r}\sum_{s=1}^{r} &\cos\left(\frac{1}{2} - \frac{2s-1}{2r}\right)(u-\pi)\sec\left(\frac{1}{2} - \frac{2s-1}{2r}\right)\pi. \end{aligned}$$

From (26) we see

$$M_2 = \frac{2}{\pi}\left(2\log\frac{4}{\pi} - \log r + \gamma\right) + \frac{1}{\pi} I,$$

where $I = \int_0^\pi (f(u) - 1)\operatorname{cosec}\frac{1}{2}u\, du$, and so to derive (6) we need show only

$$I = 2\log r. \tag{27}$$

We shall prove (27) for even r, the case of odd r following similarly with a little extra effort. Putting $v = u - \pi$, we have

$$I = \int_0^\pi \left\{\frac{2}{r}\sum_{j=1}^{(1/2)r} \frac{\cos[(2j-1)v/2r]}{\cos[(2j-1)\pi/2r]} - 1\right\}\sec\frac{v}{2}\, dv.$$

Now expanding in partial fractions we have

$$\begin{aligned} &\frac{\cos[(2j-1)v/2r]}{\cos(1/2)v} \\ &\quad = \frac{1}{r}\sum_{k=1}^{r} (-1)^{k+1}\frac{\cos[(2k-1)(2j-1)\pi/2r]\sin[(2k-1)\pi/2r]}{\cos[v/2r] - \cos[(2k-1)\pi/2r]}. \end{aligned}$$

So

$$\begin{aligned} \sec\frac{v}{2}&\sum_{j=1}^{(1/2)r}\frac{\cos[(2j-1)v/2r]}{\cos[(2j-1)\pi/2r]} \\ &= \frac{1}{r}\sum_{k=1}^{r}\frac{(-1)^{k+1}c_k\sin[(2k-1)\pi/2r]}{\cos[v/2r] - \cos[(2k-1)\pi/2r]}, \end{aligned}$$

where

$$c_k = \sum_{j=1}^{(1/2)r} \frac{\cos[(2k-1)(2j-1)\pi/2r]}{\cos[(2j-1)\pi/2r]}, \qquad k = 1,..., r.$$

Now a straightforward calculation shows

$$c_{k+1} = -c_k, \qquad k = 1,..., r-1,$$

and so $c_k = (-1)^{k+1} c_1 = (-1)^{k+1} \frac{1}{2}r$. So

$$\begin{aligned} I &= \int_0^\pi \left\{ \frac{1}{r} \sum_{k=1}^{r} \frac{\sin[(2k-1)\pi/2r]}{\cos[v/2r] - \cos[(2k-1)\pi/2r]} - \sec \frac{v}{2} \right\} dv \\ &= 2 \log r + 2 \log \sin(\pi/2r) \\ &\quad + 2 \sum_{k=2}^{r} \log \left\{ \frac{\tan[(2k-1)\pi/4r] + \tan(\pi/4r)}{\tan[(2k-1)\pi/4r] - \tan(\pi/4r)} \right\} \\ &= 2 \log r + 2 \log \sin(\pi/2r) + 2 \sum_{k=2}^{r} \log \left\{ \frac{\sin(k\pi/2r)}{\sin[(k-1)\pi/2r]} \right\} \\ &= 2 \log r. \end{aligned}$$

This completes the proof of Theorem 1.

We note that when $r = 1$, the statement of Theorem 1 requires that $\alpha_1 = 0$ if n is even and $\alpha_1 = \frac{1}{2}$ if n is odd. A modification of the above calculations produces the following result for $r = 1$ and any α_1 in$(-\frac{1}{2}, \frac{1}{2})$ if n is even, any α_1 in (0, 1) if n is odd.

If

$$\begin{aligned} \alpha &= \alpha_1, & & n \text{ even}, \\ &= \alpha_1 - \tfrac{1}{2}, & & n \text{ odd}, \end{aligned}$$

then Eq. (5) holds with $M_1 = 2/\pi$ and

$$M_2 = \frac{2}{\pi} \left\{ 2 \log \frac{4}{\pi} + \gamma + 2G(2\alpha) - G(\alpha) \right\},$$

where

$$G(x) = \sum_{k=1}^{\infty} \frac{x^2}{k(k^2 - x^2)}$$

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